REFERENCES

1. PASHKOV A.G., On one game of pursuit, PMM, 34, 5, 1970.
2. KRASOVSKII N.N. and SUBBOTIN A.I., Position differential games (Pozitsionnye differentsial'nye igry), Moscow, Nauka, 1974.
3. SUBBOTIN A.I. and CHENTSOV A.G., Approximation of the assurance in control problems (Optimizatsia garantii v zadachakh upravleniya), Moscow, Nauka, 1981.
4. KRASOVSKII N.N., Game problems of the encounter of motions (Igrovye zadachi o vstreche dvizhenii), Moscow, Nauka, 1970.
5. KRASOVSKII N.N., Differential games. Approximation and formal models, Matem. sb., 107, 4, 1978.
6. HAGEDORN P. and BREAKWELL J.V., A differential game with two Pursuers and one Evader. J. Optimization Theory and Appl., 18, 1, 1976.
7. GRIGORNENKO N.L., The pursuit of one evading object by several objects of different type, Dokl. AN SSSR, 268, 3, 1983.
8. CHIKRII A.A., Group pursuit under bounded evader coordinates, PMM, 46, 6, 1982.
9. PASHKOV A.G., On one estimate in a differential pursuit game. PMM, 36, No.6, 1972.
10. PASHKOV A.G. and TEREKHOV S.D., On a game of the optimal pursuit of an object by two other objects. PMM 47, 6, 1983.
11. PONTRYAGIN L.S., BOLTYANSKII V.G., GAMKRELIDZE R.V. and MISHCHENKO E.F., Mathematical theory of optimal processes, Moscow, Nauka, 1976.
12. BRAYSON A. and KHO YU. SHI., Applied theory of optimal control /Russian translation/, Moscow, Mir, 1972.

Translated by W.C.

PMM U.S.S.R., Vol.49, No.4, pp. 422-428, 1985 0021-8928/85 $10.00+0.00
Printed in Great Britain Pergamon Journals Ltd.

THE TRANSFORMATION OF LINEAR NON-STATIONARY OBSERVABLE AND CONTROLLABLE SYSTEMS INTO STATIONARY SYSTEMS*

N.B. VAVILOVA, V.I. KALENOVA and V.M. MOROZOV

The methodological problems of the reducibility of some classes of linear non-stationary observable and controllable systems to stationary systems is considered. The constructive use of this property to analyse the controllability and observability of non-stationary systems, and also to solve applied control and estimation problems, is proposed.

For practical applications the separation of the classes of non-stationary systems, which can be investigated using simple and effective methods similar to those for analysing stationary systems, is of interest. Linear non-stationary systems for which the fundamental matrix of the solutions can be algorithmically simply constructed using the matrix of the coefficients, pertain to these calsses; in particular systems which can be reduced to stationary systems /1–5/ using the well-known non-degenerate transformation, and also systems which are Lyapunov-reducible /6, 7/. Although for non-stationary systems the sufficient conditions for controllability and observability which do not require a knowledge of the fundamental matrix of the initial system /8–10/ are known, the search for constructive transformations which reduce the initial system to a form suitable for analysing and synthesizing simple control and estimation algorithms is important and useful.

1. Consider the linear non-stationary system

$$x^{\cdot} = A(t)\,x + B(t)\,u,\quad \sigma = C(t)\,x \tag{1.1}$$

where x is an n-dimensional state vector of the system, u is an r-dimensional vector of the controlling action, σ is a k-dimensional vector of measurements and $A(t)$, $B(t)$, $C(t)$ are matrices of corresponding dimensions, the elements of which are continuously differentiable

*Prikl.Matem.Mekhan., 49, 4, 548-555, 1985

functions of time t.

It is known that in a number of cases /1—5/ the fundamental matrix of system (1.1) can be found in explicit form using the elements of the matrix $A(t)$, in particular when the matrix $A(t)$ belongs to one of the following classes: 1) the constant matrix, 2) the diagonal matrix, 3) the triangular matrix, 4) satisfies the condition

$$A(t)\left[\int_0^t A(\tau)\,d\tau\right]=\left[\int_0^t A(\tau)\,d\tau\right]A(t) \tag{1.2}$$

5) the matrix $A_1=\mathrm{const}$ and the non-zero function $\Psi(t)$ exist, such that

$$\frac{d}{dt}\left(\frac{A(t)}{\Psi(t)}\right)=A_1A(t)-A(t)A_1$$

if $\Psi(t)\equiv 1$, then $A(t)$ obeys the equation

$$A^{\cdot}(t)=A_1A(t)-A(t)A_1 \tag{1.3}$$

Other cases, including more general ones, in which the matrix $A(t)$ is subject to fairly complex conditions, are presented in /1, 2, 5/. Note that one special class of matrices $A(t)$ - which satisfy condition (1.2), considered in /3/ - was previously investigated in a more general form /2/.

We will consider systems of the form (1.1), in which the fundamental matrix can be found in closed form. We shall determine the conditions for reducing the non-stationary system (1.1) to a stationary system.

2. Suppose $\Phi(t)$ is the fundamental matrix of the system, corresponding to the first equation (1.1) ($\Phi(t_0)=E$). Then the linear transformation

$$x=\Phi(t)y \tag{2.1}$$

reduces system (1.1) to the form

$$y^{\cdot}=N(t)u,\quad \sigma=H(t)y \tag{2.2}$$
$$(H(t)=C(t)\Phi(t),\quad N(t)=\Phi^{-1}(t)B(t))$$

Theorem 1. Using a linear non-degenerate transformation we can reduce (2.2) to a completely stationary system of the same order

$$z^{\cdot}=Rz-Mu,\ \sigma=Lz \tag{2.3}$$

(containing the matrices $z\,(n\times 1)$, $R\,(n\times n)$, $M\,(n\times r)$, $L\,(k\times n)$) only when the matrices $H(t)$, $N(t)$ satisfy the equations

$$H^{\cdot}=HG,\quad N^{\cdot}=-GN \tag{2.4}$$

($G\,(n\times n)$ is a constant matrix). At the same time the matrices R, L and M in system (2.3) are determined by the relations

$$R=G,\ L=H(t_0),\ M=N(t_0)$$

Proof. Sufficiency. The transformation $y=\exp(-G(t-t_0))z$ reduces (2.2) to the form

$$z^{\cdot}=Gz-N(t_0)u,\ \sigma=H(t_0)z$$

Necessity. Suppose the transformation $y=Qz$ reduces (2.2) to the form (2.3). Then the matrix $Q(t)$ satisfies the equation $Q^{\cdot}=-QR$ ($Q(t_0)=E$). Differentiating the equations $H(t)Q(t)=L$, $N(t)=Q(t)M$ we obtain $H^{\cdot}=HR$, $N^{\cdot}=-RN$.

For the first equation (1.1) with matrix $A(t)$ of the above type, besides the transformation (2.1) other linear non-degenerate transformations also exist, reducing (1.1) to the form

$$x^{\cdot\prime}=A'x'+B'(t)u,\ \sigma=C'(t)x';\ A'=\mathrm{const} \tag{2.5}$$

In particular, if the matrix $A(t)$ satisfies condition (1.3), the transformation $x=\exp(A_1(t-t_0))x'$ reduces the first equation (1.1) to the form (2.5), where $A'=A(t_0)-A_1$.

Note that any system (1.1), in which a system corresponding to the first equation (1.1) when $B(t)\equiv 0$ is reducible in Lyapunov's sense /6, 7/ can be transformed to the form (2.5).

Consider system (2.5). The transformation $x'=\exp(A'(t-t_0))y$ reduces this system to the form (2.2), where

$$H(t)=C'(t)\exp(A'(t-t_0)),\ N(t)=\exp(-A'(t-t_0))B'(t)$$

Then from Theorem 1 we have

Theorem 2. System (2.5) reduces to the stationary system (2.3) only when the matrices $C'(t), B'(t)$ satisfy the equations

$$C^{\prime\cdot}=C'[-A'+\exp(A'(t-t_0))G\exp(-A'(t-t_0))] \tag{2.6}$$

$$B'^{\cdot} = [A' - \exp(A'(t - t_0)\, G \exp(-A'(t - t_0))]\, B'$$

($G\,(n \times n)$ is some constant matrix).

In particular, if the matrix G is such that the condition $A'G = GA'$ holds, then $C'(t)$ and $B'(t)$ satisfy the equations

$$C'^{\cdot} = C'(-A' + G), \quad B'^{\cdot} = (A' - G)\, B' \tag{2.7}$$

Corollary. Suppose the matrices $C'(t)$ and $B'(t)$ obey the equations $C'^{\cdot} = C'K$, $B'^{\cdot} = -KB'$, where K is a constant matrix and $KA' = A'K$, then the change of variables $x' = \exp(-K(t - t_0))\, z$ reduces (2.5) to the stationary system (2.3), where $R = A' + K$, $M = B'(t_0)$, $L = C'(t_0)$.

Note that it is more convenient, without verifying the validity of condition (2.6), to construct the matrices $H(t)$, $N(t)$ and then verify that conditions (2.4) hold.

3. The theorems formulated in Sect.2 give the necessary and sufficient conditions for the possibility of reducing the non-stationary system of the type considered to a stationary system without increasing the order of the initial system.

Consider the more general case when the non-stationary system (1.1) can be transformed into a completely stationary system using a change of variables which expands the state space.

We will further assume that in the first equation (1.1) $B(t) \equiv 0$. Suppose it is reduced to the form (A_y is a constant matrix)

$$y^{\cdot} = A_y y \tag{3.1}$$

using the transformation

$$x = T(t)\, y, \quad \det T(t) \neq 0, \quad \forall t \geqslant t_0$$

The equation of measurements (the second equation (1.1)) after changing to the new variables has the form

$$\sigma = C(t)\, T(t)\, y = H_1(t)\, y \tag{3.2}$$

(when $T(t) = \Phi(t)$, $A_y = 0$, $H_1(t) = H(t)$)

For simplicity, we shall assume that the matrix $C(t)$ is a $1 \times n$ matrix, then $H_1(t) = h^T(t)$, where $h(t)\ (n \times 1)$ is a column-matrix.

We will assume that $h^T(t)$ can be represented in the form

$$h^T(t) = f^T(t)\, D \tag{3.3}$$

where D is some constant $m \times n$ matrix and $f(t)$ is a $m \times 1$ $(m \geqslant n)$ vector-function of time satisfying the equation

$$f^{\cdot} = Sf, \quad S\,(m \times m) = \mathrm{const}. \tag{3.4}$$

This means, in particular, that functions such as polynomials, exponential functions, finite sums of trigonometric functions, etc., belong to the class of functions $h_i(t)$ $(i = 1, 2, \ldots, n)$ which are components of the vector $h(t)$ and are separated in this way.

The vector-function $h^T(t)$, defined by Eq.(3.3), in the general case does not satisfy an equation of the form (2.7), and for system (3.1)–(3.4) the conditions for reducing it to a stationary system of the same order do not hold. In particular, if $D = E$ and $h(t)$ obeys Eq.(3.4), it is necessary that $A_y S^T = S^T A_y$.

It is shown in /11/ (see also /12/), that, using the transformation

$$q = \Sigma(t)\, y; \quad \Sigma(t) = E_n \otimes f(t) \tag{3.5}$$

where $\Sigma(t)$ is an $mn \times n$ matrix, q is an $mn \times 1$ vector, $E_n\,(n \times n)$ is a unit matrix and the symbol $\otimes$ denotes the Kronecker product of the matrices /13/, (1.1) can be reduced to the form

$$q^{\cdot} = A_q q, \quad \sigma = d^T q \quad (A_q = E_n \otimes S + A_y \otimes E_m) \tag{3.6}$$

(the vector $d\,(mn \times 1)$ is formed from the successively written columns of the matrix D).

Thus, (1.1) reduces to the stationary system (3.6) - but of higher dimensions - when the above conditions hold.

It was shown /11/ that for such a reduction of the observation-non-stationary system (3.1)–(3.4) to the completely stationary system (3.6) the property of observability of the initial system is preserved. We can show that if the initial system is observable, then a closed n-th order system to determine the initial variables y is separated from the expanded system (3.6).

4. Consider the problem of constructing an estimate of the state vector x of systems (3.1), (3.2). We will assume that (3.6) and, consequently, (3.1), (3.2) are observable. The estimation of the state vector is not completely the observable system considered in /12/.

When constructing an estimate of the vector x we will proceed from the stationary system (3.6). The estimation algorithm has the form

$$q^{\cdot c} = A_q q^c + K(\sigma - d^T q^c), \quad q^c(t_0) = 0 \tag{4.1}$$

Since (3.7) is observable then, as is known /14/, the vector K in Eq.(4.1) can be chosen as a constant guaranteeing any degree of attenuation of the error of the estimate $\Delta q = q - q^\circ$

$$\| \Delta q (t) \| \leqslant \gamma_1 \exp (\lambda_1 t)$$

where $\gamma_1 = \text{const} > 0$, and $\lambda_1 < 0$ is any number specified in advance.

We will show that the vector K can be chosen so as to guarantee any degree of attenuation of the error of the estimate of the state vector of the initial non-stationary system $\Delta y = y - y^\circ$.

The estimates q° and y° of the vectors q and y and the corresponding errors of the estimates Δq and Δy are connected, by virtue of (3.5), with the relations

$$q^\circ (t) = \Sigma (t) y^\circ (t), \quad \Delta q (t) = \Sigma (t) \Delta y (t) \tag{4.2}$$

which are redefined sets of linear algebraic equations in the components of the vectors $y^\circ (t)$ and $\Delta y (t)$.

We shall write the solutions of (4.2) in the form

$$y^\circ (t) = P (t) q^\circ (t), \quad \Delta y (t) = P (t) \Delta q (t) \tag{4.3}$$

where $P (t)$ is a matrix satisfying the equation

$$P (t) \Sigma (t) = E_n \tag{4.4}$$

(We can, for example, take the pseudoinverse matrix $\Sigma^+ (t)$ as the matrix $P (t)$.)

By virtue of the second relation (3.5) the elements of the matrix $\Sigma (t)$ are solutions of the linear system with constant coefficients (3.4). Bearing in mind relation (4.4), we will have $\| P (t) \| \leqslant \gamma_2 \exp (\lambda_2 t)$, where $\gamma_2 > 0$ and λ_2 are certain constants. Then from relations (4.3) it follows that

$$\| \Delta y (t) \| \leqslant \| P (t) \| \| \Delta q (t) \| \leqslant \gamma_1 \gamma_2 \exp ((\lambda_1 + \lambda_2) t)$$

Choosing $\lambda_1 < \lambda_0 - \lambda_2$, we can obtain any degree of attenuation $\lambda_0 < 0$ of the error of the estimate $\Delta y (t)$.

We can directly obtain the estimate $y^\circ (t)$ as the solution of the differential equation

$$y^{\cdot\circ} = A_y y^\circ + P (t) K (\sigma - h^T (t) y^\circ)$$

5. As is well-known /8, 10, 14/, the fact of the duality of problems of estimating state and control problems exists, consisting of the following statement: the system $x^\cdot = A (t) x + B (t) u$ is only controllable when the system $x^\cdot = -A^T (t) x, z = B^T (t) x$, conjugate to it, is observable. In this connection, the results obtained in /11/ can be useful for investigating linear systems which are non-stationary with resepct to control and have a non-stationarity of a definite form.

Consider the linear system

$$\xi^\cdot = A \xi + D^T f (t) u \tag{5.1}$$

where ξ is an n-dimensional vector describing the state of the system, A is a constant $(n \times n)$ matrix, $D^T (n \times m)$ is a matrix with constant elements, u is a scalar control, and $f (t)$ is a known $m \times 1$ vector-function of time, which satisfies, as previously, the set of linear equations with constant coefficients (3.4).

We shall determine for system (5.1), (3.4) the stationary system corresponding to it, the variables of which are connected with the initial variables by known linear relations. We shall take a linear system conjugate to (5.1)

$$x^\cdot = -A^T x, \quad z = f^T (t) D x \tag{5.2}$$

As shown above, using the change of variables (3.5), system (5.1) can be reduced to the stationary system

$$y^\cdot = Q^T y, \quad z = d^T y \tag{5.3}$$

The matrix Q^T is determined in a similar way to (3.6) using the formula

$$Q^T = - A^T \otimes E_m + E_n \otimes S \tag{5.4}$$

and the raw d^T is formed form the successively located rows of matrix D:

$$d^T = (d_{11} d_{12} \ldots d_{1m} \ldots d_{n1} d_{n2} \ldots d_{nm})$$

The stationary system, conjugate to (5.3), has the form

$$\eta^\cdot = -Q \eta + d u \tag{5.5}$$

We can show that the variables ξ and η are connected by the relation ($\Sigma^T (t)$ is an $n \times mn$ matrix)

$$\xi = \Sigma^T (t) \eta, \quad \Sigma^T (t) = E_n \otimes f^T (t) \tag{5.6}$$

Thus, the transformation (5.6), in which η satisfies system (5.5) with the initial conditions $\eta(t_0)=\eta_0$, enables one to obtain a state vector ξ, whose behaviour is described by the set of equations (5.1) with the initial conditions $\xi(t_0)=\Sigma^T(t_0)\eta(t_0)$.

Since the rank of the matrix $\Sigma^T(t_0)$ equals n, we can represent the solution of system (5.1) in the form (5.6), where η is a solution of the stationary system (5.5).

Note that the controllability of (5.5) is only a sufficient condition for the controllability of (5.1).

Suppose the transformation $v=My$, where M is the matrix $l\times mn$, separates the observable subspace $\{v\}$ from the space of the states $\{y\}$. The vector v obeys the equation

$$v^{\cdot}=Q_1^Tv,\quad z=d_1^Tv \tag{5.7}$$

The matrices Q_1^T and d_1^T, representing an observable pair, satisfy the relations $MQ^T=Q_1^TM,\ d^T=d_1^TM$. The variables x and v are connected by the linear transformations

$$v=M\Sigma(t)x \tag{5.8}$$

and, as shown in /11/, if (5.2) is observable, then $\operatorname{rank} M\Sigma(t)=n$.

We will write the controllable system, conjugate to (5.7)

$$\xi^{\cdot}=-Q_1\xi+d_1u \tag{5.9}$$

We can show that the vector $\eta=M^T\xi$, where ξ obeys Eq.(5.9) with the initial condition $\xi(t_0)=\xi_0$, will satisfy Eq.(5.5) with the initial condition $\eta(t_0)=M^T\xi_0$.

Summing up the above concerning the initial system (5.1) and the stationary systems (5.5) and (5.8), we obtain that the vector

$$\zeta=\Sigma^T(t)M^T\xi \tag{5.10}$$

where the vector ξ is a solution of the controllable stationary system (5.9) with the initial condition $\xi(t_0)=\xi_0$, will satisfy Eq.(5.1) with the initial condition $\zeta(t_0)=\Sigma^T(t_0)M^T\xi_0$, and if (5.1) is controllable, then according to (5.8) $\operatorname{rank}\Sigma^T(t_0)M^T=n$.

Hence, the above method of reducing non-stationary controllable systems of the class considered to stationary systems enables us to reduce the solution of different problems of controlling such non-stationary systems, to corresponding problems for stationary systems, the methods of solving which are well developed.

6. The technique presented can be used to solve a fairly wide class of control and estimation problems.

Suppose the mechanical system permits stationary rotation around some fixed axis in space. In many cases the equations in variations for this stationary motion in the fixed system of coordinates are a set of linear differential equations with periodic coefficients.

When controlling (estimating) these systems roblems often arise in which the controlling influences (measurements) are formed in a fixed set of coordinates. Then the equations of the controllable object in a fixed set of coordinates have the form

$$\xi^{\cdot}=P(t)\xi+B_1u,\ P(t+T_P)=P(t)\ (\sigma=H_1\xi) \tag{6.1}$$

Here σ is the vector of measurements and B_1 and H_1 are constant matrices.

In a fixed set of coordinates, connected with the rotating object, the equations of motion can be represented in the form (A is a constant matrix)

$$\begin{gathered}x^{\cdot}=Ax+B(t)u\quad(\sigma=H(t)x)\\ B(t+T_P)=B(t),\ H(t+T_P)=H(t)\end{gathered} \tag{6.2}$$

System (6.2) belongs to the type of system (5.1) or (3.1)–(3.4).

Example. 1°. The equations of motion of a rigid body, stabilized by means of its own rotation, with motors that are rigidly attached to the body, have the form (6.2) /15/

$$x_1^{\cdot}=-ax_2+u\cos\Omega_1t,\ x_2^{\cdot}=ax_1-u\sin\Omega_1t \tag{6.3}$$

Here x_1, x_2 are projections of the angular velocity of the body on an axis, rigidly connected with it and orthogonal to the axis of the rotation, Ω_1 is the angular velocity of rotation, u is the controlling factor, and a is some constant.

The change of variables of the type (5.6)

$$x_1=y_1\sin\Omega_1t+y_2\cos\Omega_1t,\ x_2=y_3\sin\Omega_1t+y_4\cos\Omega_1t$$

reduces (6.3) to the stationary system

$$\begin{gathered}y_1^{\cdot}=\Omega_1y_2-ay_3,\ y_2^{\cdot}=-\Omega_1y_1-ay_4+u\\ y_3^{\cdot}=\Omega_1y_4+ay_1-u,\ y_4^{\cdot}=-\Omega_1y_3+ay_2\end{gathered} \tag{6.4}$$

System (6.4) is non-controllable. From it the following controlling subsystem is easily separated:

$$z_1^{\cdot} = -(a + \Omega_1) z_2 + u, \quad z_2^{\cdot} = (a + \Omega_1) z_1$$
$$(z_1 = y_2 - y_3, \; z_2 = y_1 + y_4)$$

In Eq.(5.10) the initial variables x_1, x_2 are connected with the controllable variables z_1, z_2 by the transformation

$$x_1 = z_1 \cos \Omega_1 t + z_2 \sin \Omega_1 t, \; x_2 = -z_1 \sin \Omega_1 t + z_2 \cos \Omega_1 t$$

2°. The linearized equations of the perturbed motion of the material point with a circular orbit of radius r_1 in a central field of forces in a fixed system of coordinates have the form

$$x^{\cdot} = A(\tau) x; \quad A(\tau) = \left\| \begin{matrix} O_2 & E_2 \\ F & O_2 \end{matrix} \right\|, \quad F = \left\| \begin{matrix} a_+ & b \\ b & a_- \end{matrix} \right\| \tag{6.5}$$
$$a_{\pm} = {}^1/_2 (1 \pm 3 \cos 2\tau), \; b = {}^3/_2 \sin 2\tau, \; \tau = \omega_1 t$$

(ω_1 is the angular velocity of the rotation, O_2 is the zero matrix, and E_2 is a 2×2) unit matrix).

The equation of measurements in the case when the known distance from the point considered to another point, moving in the same plane along a known angular orbit of radius r_2 with the angular velocity ω_2, has the form

$$\sigma = (\cos \tau - \rho \cos n_1 \tau) x_1 + (\sin \tau - \rho \sin n_1 \tau) x_2 \tag{6.6}$$
$$(n_1 = \omega_2 / \omega_1, \; \rho = r_2 / r_1)$$

where x_1, x_2 are the perturbations of the vector radius and the polar angle. The matrix $A(\tau)$ is related to the type of matrix whose behaviour is described by Eq.(1.4) with matrix A_1 of the form

$$A_1 = \left\| \begin{matrix} \chi & O_2 \\ O_2 & \chi \end{matrix} \right\|, \quad \chi = \left\| \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \right\|$$

According to /4/, the change of variables $x = \exp(A_1 t) y$ reduces (6.5) to the stationary system

$$y^{\cdot} = A_2 y \tag{6.7}$$
$$A_2 = A(0) - A_1 = \left\| \begin{matrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{matrix} \right\|$$

The equations of measurements (6.6) in the variables y will take the form

$$\sigma = h^T y, \; h^T = (1 - \rho \cos \nu\tau, \; \rho \sin \nu\tau, \; 0, \; 0), \; \nu = n_1 - 1 \tag{6.8}$$

We can show that the vector $h(\tau)$ does not satisfy an equation of the type (2.6), but it is represented in the form (3.3), (3.4). As in the transformation (3.5) we will introduce the variables

$$q_k = y_k \exp(i\nu\tau) \; (k = 1, 2, 3, 4)$$

In these variables system (6.7), (6.8) becomes completely stationary

$$\begin{aligned} q_1^{\cdot} &= q_2 + q_3 + i\nu q_1, \quad y_1^{\cdot} = y_2 - y_3 \\ q_2^{\cdot} &= -q_1 + q_4 - i\nu q_2, \quad y_2^{\cdot} = -y_1 + y_4 \\ q_3^{\cdot} &= 2q_1 + q_4 + i\nu q_3, \quad y_3^{\cdot} = 2y_1 + y_4 \\ q_4^{\cdot} &= -q_2 - q_3 + i\nu q_4, \quad y_4^{\cdot} = -y_2 - y_3 \end{aligned} \tag{6.9}$$

$$\sigma = y_1 - \rho \operatorname{Re} q_1 + \rho \operatorname{Im} q_2 \tag{6.10}$$

Note that when analysing the observability of (6.9) we should take into consideration the equations for the conjugate variables $\bar{q}_k$ $(k = 1, 2, 3, 4)$ and should represent the equation of measurements (6.10) in the form

$$\sigma = y_1 - \rho \frac{q_1 + \bar{q}_1}{2} - \rho \frac{q_2 - \bar{q}_2}{2} i$$

The procedure used in Example 2° was used /16/ in the problem of correcting an intertial navigation system using additional data on the distance from the object to an artificial navigation satellite. Note, also, that equations of type (6.1) occur in the problem of correcting inertial navigation systems in the case when the object banks correctly /17/.

REFERENCES

1. ERUGIN N.P., Linear sets of ordinary differential equations with periodic and quasiperiodic coefficients. Minsk: Izd-vo AN BSSR, 1963.
2. SALAKHOVA I.M. and CHEBOTAREV G.N., The solvability in finite form of some sets of linear differential equations. Izv. vuzov. Matematika, 3, 1960.
3. WU M.-Y and SHERIF A., On the commutative class of linear time-varying systems. Internat. J. Control, 23, 3, 1976.

4. WU M.-Y., Transformation of a linear time-varying system into a linear time-invariant system. Internat. J. Control, 27, 4, 1978.
5. WU M.-Y., A successive decomposition method for the solution of linear time-varying systems. Internat. J. Control, 33, 1, 1981.
6. LYAPUNOV A.M., The general problem of the stability of motion. Moscow-Leningrad: Gostekhizdat, 1950.
7. ERUGIN N.P., Reducible systems. Tr. Matem. in-ta im. V.A. Steklov, 13, 1946.
8. KRASOVSKII N.N., Theory of motion control. Moscow: Nauka, 1968.
9. SILVERMAN L.M. and MEADOWS H.E., Controllability and observability in time-variable linear systems. SIAM J. Control, 5, 1, 1967.
10. D'ANGELO G., Linear systems with variables parameters. Moscow: Mashinostroenie, 1974.
11. MINTS N.B., MOROZOV V.M. and UKRAINTSEV S.V., Estimation of the state vector of linear systems which are non-stationary under observation. In: Some problems of the theory of navigation systems. Moscow: Izd-vo MGU, 1979.
12. PARUSNIKOV N.A., MOROZOV V.M. and BORZOV V.I., The correction problem in inertial navigation. Moscow: Izd-vo MGU, 1982.
13. BELLMAN R., Introduction to matrix theory. Moscow: Nauka, 1969.
14. KALMAN R., FALB P. and ARBIB M., Outlines of the mathematical theory of systems. Moscow: Mir, 1971.
15. LUNDERSHTADT R., A system for controlling a space apparatus - stabilized by its own rotation - which is optimal in speed of response and fuel consumption. In: Control in space. 2, Moscow: Nauka, 1973.
16. KALENOVA V.I. and MOROZOV V.M., Observability in the problem of correcting inertial navigation systems using additional dtaa from an aritificial satallite, Kosmich. issledovaniya, 22, 3, 1984.
17. MOROZOV V.M., MATASOV A.I. and SHAKOT'KO A.G., The observability of the parameters of an inertial navigation system in correct banking. Izv. AN SSSR. MTT, 4, 1982.

Translated by H.Z.

PMM U.S.S.R.,Vol.49,No.4,pp.428-433,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

FLUCTUATION HYDRODYNAMICS OF THE BROWNIAN MOTION OF A PARTICLE IN A FIXED DISPERSED LAYER*

A.G. BASHKIROV

The influence of the perturbation exerted by a grid of fixed spherical particles, randomly distributed in space, on the Brownian diffusion of particles suspended in the flow of a fluid which penetrates the grid is disucssed. The fixed particles affect the coefficient of diffusion that is transverse to the flow in two ways: on the one hand they reduce it in accordance with the Stokes coefficient, and on the other they increase it because of the influence of a random velocity field which is generated by the flow past the randomly distributed particles. A convective diffusion equation is derived on the basis of the Fokker-Planck equation for a distribution function. A stochastic diffusion equation (of Langevin's type) obtained with a random velocity field is solved by the method of Green's function, whence the desired diffusion coefficient is found. The errors allowed when solving a similar problem in /1/ are indicated.

The fluctuation hydrodynamics of Brownian motion in a homogeneous viscous fluid was discussed in /2/ where, in particular, an expression for the coefficient of the particle resistance was obtained in terms of the fluctuation characteristics of the fluid. Later, the influence of hydrodynamic fluctuations on the diffusion of a particle in a homogeneous fluid was examined in /3/: it was shown that the diffusion coefficient of a particle that is large with respect to intermolecular distances is determined entirely by the thermal fluctuations of the fluid velocity field. This result was also confirmed by the microscope kinetic theory of Brownian motion in /4, 5/, where an expression similar to Kubo's formula, for the coefficient of resistance of a large particle in terms of the fluctuation

**Prikl.Matem.Mekhan.*,49,4,556-562,1985